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THE COMBINATORICS OF *n*-CATEGORICAL PASTING*

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The purpose of this paper is to prove the n-category pasting theorem. The theorem, which asserts that the categorical operation of pasting is well defined, has not previously been proved, mainly because of the lack of a sufficient formalization of the diagrams on which pasting operates. The paper develops a combinatorial treatment of these diagrams and proves the pasting theorem.

Introduction

With the rise in interest in n-categories, at least for $n = 2$, the operation of *pasting* has been recognized as a valuable tool for working with several different compositions. Typically, pasting is used to specify a cell by giving a *pasting diagram* (see e.g. (1) below). The pasting theorem says that such a cell is well defined – the several different sequences of compositions which the diagram could be interpreted as representing yield the same cell.

Pasting was introduced by Bénabou in his treatment of bicategories [2]. Walters in his thesis [12] used pasting and in a 1971 lecture in Sydney [13] he emphasized the importance of the pasting theorem, which he stated as part of an alternative axiomatics for 2-categories. Later, pasting played an important role in the joint work of Street and Walters [11].

An expository description of pasting can be found in the review of Kelly and Street [6] which also includes a description of the pasting theorem. Despite a suggested technique, and a number of attempts, no proof of the theorem has appeared. It seems that the theorem has never been proved, chiefly because of the lack of a sufficient formalization of the notion of pasting diagram. This paper introduces a formal theory of pasting diagrams and investigates their properties. Once the appropriate diagrams have been isolated, a combinatorial analysis shows that certain ω-categories constructed from the diagrams are *free* ω-categories and the pasting theorem follows as an easy corollary (Observation 15).

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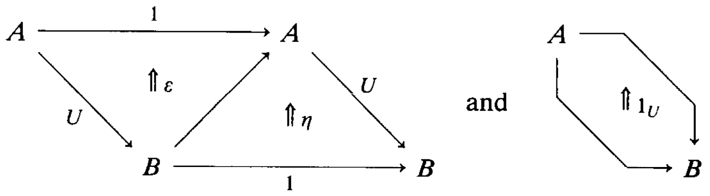
The results reported here were described to the Sydney Category Seminar in 1966 and the Louvain-La-Neuve International Category Theory Meeting in 1987. Since then, Power has developed a graph-theoretic treatment of the 2-category case [7] and has produced a preliminary version of the 3-category case. Schanuel [8] has begun yet another formulation of the 2-category case.

The author intends to write a modified and more conceptual version of the 2-category case, but the general n-category case is presented here because it is required for applications to computer science and to coherence theorems which will be pursued in subsequent papers.

1. Diagrams

In category theory, diagrams are used in at least two senses: to present the data for the calculation of some limit or colimit, and to define a cell as a composite of other cells. In what follows we deal exclusively with the latter usage.

A diagram in a category has been defined to be a graph morphism from some graph into the underlying graph of the category [1]. In 2-category theory, pasting diagrams like

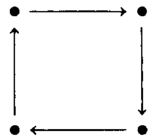


(1)

(the equality of which expresses one of the triangular equations of an adjunction) play an important role. If a diagram in a 2-category were to be a 2-graph morphism [3] from some 2-graph into the underlying 2-graph of the 2-category, then the left-hand side of (1) would not be a diagram (although the right-hand side would be a diagram).

Street [9], recognizing this difficulty, introduced *computads*. A computad $\mathscr{G}$ is a graph $|\mathscr{G}|$ together with a second graph structure whose edges are called 2-cells and whose vertices are elements of the free category on $|\mathscr{G}|$. Futhermore, in the second graph structure, two vertices can be connected by an edge only if they share the same domain and codomain as elements of the free category on $|\mathscr{G}|$. Street defined the underlying computad of a 2-category in which a 2-cell from the 2-category appears between every possible factorization of its domain and codomain. A diagram in a 2-category may be taken to be a computad morphism into the underlying computad of the 2-category.

We take the view that the above use of graphs and computads defines a diagram by a *parametrization*. However, a parametrizing object is usually in some sense 'loop free' or 'non-singular' and there is no such requirement above. So, for example, the definitions allow



to occur as a parametrizing object and hence a similar square of morphisms forms a diagram in a category. It is not at all clear what the composite of such a diagram should be.

In Section 2 we introduce *pasting schemes.* A *loop-free pasting scheme* is an appropriate parametrizing object for a diagram in an n-category. A *realization* of a loop-free pasting scheme in a particular n-category C is the map which defines the parametrization and is given by specifying a k-cell of C for each k-dimensional element of the pasting scheme. We call a realization *appropriate* if it respects domains and codomains. A *well-formed pasting scheme* is the parametrizing object for a composable diagram – sometimes referred to as a 'leg' in a diagram in an ordinary category. The *n-category pasting theorem* states that a well-formed pasting scheme with a given appropriate realization, determines a unique composite as follows.

Suppose that A is a well-formed loop-free pasting scheme and that r is an appropriate realization of A in some ω-category C. We will see that the collection of well-formed subpasting schemes of A forms an ω-category $\mathscr{P}(A)$ (Theorem 12) and that the freeness of this ω-category (Theorem 13) gives a bijection between appropriate realizations of A in C and ω-functors from $\mathscr{P}(A)$ to C. Finally, A itself is a cell in $\mathscr{P}(A)$ and its image under the ω-functor corresponding to r is the composite of the diagram specified by A and r.

2. Pasting schemes

In this section we set down the technical details needed in order to be precise about diagrams like (1) above. Such a diagram will be determined by a realization of a pasting scheme in a category. A pasting scheme will be a graded set $(A_i)_{i\in\omega}$, where for each i, A_i represents the set of i-cells in the diagram. The actual arrangement of the cells relative to one another will be determined by two collections of relations $\mathsf{E}^i_j, \mathsf{B}^i_j : A_i \to A_j$ which may be thought of as describing which j-cells are at the 'end', respectively 'beginning', of each of the i-cells.

Let $A=(A_i)_{i\in\omega}$ be a graded set, E^i_j, $i,j\in\omega$, $j\le i$, a collection of relations with E^i_j a relation between the sets A_i and A_j. Let X be a subgraded set of A of dimension n. Write $\mathsf{E}^k(X)_i=\{y\in A_i$: there exists $x\in X_k$, $x\mathsf{E}^k_i y\}$ and $\mathsf{E}(X)$ for $\mathsf{E}^n(X)$. If E^i_j, B^i_j are two such collections of relations, let R^i_j be the relation between A_i and A_j given by $x \mathsf{R}^i_j y$ when there exists a sequence $x = x_1, x_2, \ldots, x_j = y$ of elements of A satisfying $x_k \mathsf{D}^p_q x_{k+1}$ for $k = 1, 2, \ldots, j-1$ and $\mathsf{D}^p_q = \mathsf{E}^p_q$ or B^p_q.

We will often position the grading subscript on the relation writing $\mathsf{E}_i(X)$ rather than $\mathsf{E}(X)_i$. The relation E^i_j is called *finitary* when, for any $x \in A_i$, $\mathsf{E}^i_j(x)$ is finite.

In what follows, the E^i_j will be 'end' relations and the B^i_j 'beginning' relations and we have a *duality*: If P is a proposition, then $dual_k P$ stands for the proposition obtained from P be replacing all occurrences of E^k by B^k and vice versa.

A *pasting scheme* $(A, \mathsf{E}, \mathsf{B})$ is a graded set (A_i) together with finitary relations $\mathsf{E}^i_j, \mathsf{B}^i_j$, $j \le i$, such that

(i) E^i_j is a relation between A_i and A_j;

(ii) E^i_i is the identity relation on A_i;

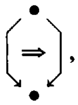
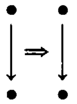
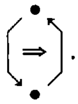
(iii) for $k > 0$ and any $x \in A_k$ there exists $y \in A_{k-1}$ with $x \mathsf{E}^k_{k-1} y$;

(iv) for $k < n$, $w \mathsf{E}^n_k x$ if and only if there exists u, υ such that $w \mathsf{E}^n_{n-1} u \mathsf{E}^{n-1}_k x$ and $w \mathsf{E}^n_{n-1} \upsilon \mathsf{B}^{n-1}_k x$;

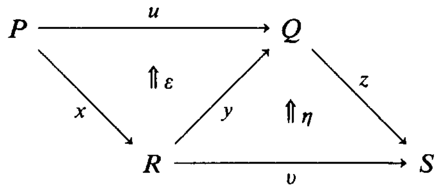
(v) if $w \mathsf{E}^n_{n-1} z \mathsf{E}^{n-1}_k x$, then either $w \mathsf{E}^n_k x$ or there is a υ with $w \mathsf{B}^n_{n-1} \upsilon \mathsf{E}^{n-1}_k x$ and dually (notice that there are four dual forms of condition (v)).

We will allow A to ambiguously denote either the pasting scheme or its graded set.

Informally, condition (iii) says that every k-cell ends at at least one $k-1$ cell, and dually begins at at least one $k-1$ cell. Condition (iv) ensures that low dimensional ends occur between higher dimensional ends – see for instance $Q \in \mathsf{E}(\eta)$ in Example 1 below. Finally, condition (v) ensures that a cell's beginnings and ends 'close up' and that their orientations agree:

, rather than  or .

Example 1. The diagram



is a representation of the pasting scheme $(A, \mathsf{E}, \mathsf{B})$ given by

$$A_0 = \{P, Q, R, S\}, \qquad A_1 = \{u, \upsilon, x, y, z\},$$
$$A_2 = \{\varepsilon, \eta\}, \qquad A_k = \emptyset, \quad k > 2,$$

$\mathsf{E}^2_2 = \{(\varepsilon, \varepsilon), (\eta, \eta)\}$ $\qquad$ $\mathsf{B}^2_2 = \{(\varepsilon, \varepsilon), (\eta, \eta)\}$

$\mathsf{E}^2_1 = \{(\varepsilon, u), (\eta, y), (\eta, z)\}$ $\qquad$ $\mathsf{B}^2_1 = \{(\varepsilon, x), (\varepsilon, y), (\eta, \upsilon)\}$

$\mathsf{E}^2_0 = \{(\eta, Q)\}$ $\qquad$ $\mathsf{B}^2_0 = \{(\varepsilon, R)\}$

$$\mathsf{E}_1^1 = \{(u,u),(v,v),(x,x),(y,y),(z,z)\} \qquad \mathsf{B}_1^1 = \{(u,u),(v,v),(x,x),(y,y),(z,z)\}$$
$$\mathsf{E}_0^1 = \{(u,Q),(v,S),(x,R),(y,Q),(z,S)\} \qquad \mathsf{B}_0^1 = \{(u,P),(v,R),(x,P),(y,R),(z,Q)\}$$
$$\mathsf{E}_0^0 = \{(P,P),(Q,Q),(R,R),(S,S)\} \qquad \mathsf{B}_0^0 = \{(P,P),(Q,Q),(R,R),(S,S)\}$$

In a pasting scheme A, define $\lhd_A$ (written as $\lhd$ when there is no danger of confusion) as follows: for any k, and for any $a,b \in A_k$, say $a \lhd b$ if there is a sequence

$$a = a_0, a_1, \ldots, a_j = b, \quad j > 0,$$

of elements of A_k with, for all $i<j$, $\mathsf{E}_{k-1}(a_i) \cap \mathsf{B}_{k-1}(a_{i+1}) \neq \emptyset$. As usual, if X is a subgraded set of A, we write $\lhd^k(X)$ for $\{b \in A$: there exists $x \in X_k$, $b \lhd x\}$, and if X is n-dimensional, $\lhd(X)$ for $\lhd^n(X)$.

A pasting scheme A is said to have *no direct loops* when, for any k and for any $a,b \in A_k$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$ and $a \lhd b$ implies $\mathsf{B}(a) \cap \mathsf{E}(b) = \emptyset$.

If A is a pasting scheme and X a finite subgraded set of A, define the *domain* of X, $\operatorname{dom} X$ by $X - \mathsf{E}(X)$ and the *codomain* of X, $\operatorname{cod} X$ by $X - \mathsf{B}(X)$.

Lemma 2. *If A is a finite, k-dimensional pasting scheme with no direct loops, then* $\operatorname{dom} A$ *is a $(k-1)$-dimensional graded set.*

Proof. The domain of A is at most $(k-1)$-dimensional since $(\operatorname{dom} A)_k = A_k - \mathsf{E}_k(A_k) = \emptyset$. To see that $\operatorname{dom} A$ is $(k-1)$-dimensional choose some $a_0 \in A_k$ and some $y_0 \in \mathsf{B}_{k-1}(a_0)$. If $y_0 \notin \operatorname{dom} A$, it can only be because $y_0 \in \mathsf{E}(a_1)$ for some $a_1 \in A_k$. Now choose any $y_1 \in \mathsf{B}_{k-1}(a_1)$, and repeat. Since A has no direct loops, $a_i \neq a_j$ for $i \neq j$ and so, since A_k is finite, we must eventually locate a $y_m \in (\operatorname{dom} A)_{k-1}$. □

Theorem 3. *If A is a finite pasting scheme with no direct loops, then*

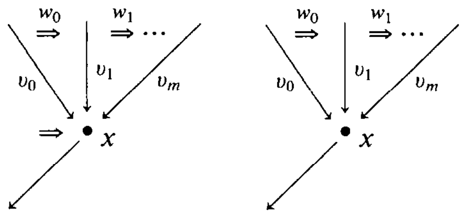
$$\operatorname{dom}\operatorname{dom} A = \operatorname{dom}\operatorname{cod} A.$$

Proof. Notice that

$$\operatorname{dom}\operatorname{dom} A = (A - \mathsf{E}(A)) - \mathsf{E}(A - \mathsf{E}(A)) = A - (\mathsf{E}(A) \cup \mathsf{E}(A - \mathsf{E}(A))),$$
$$\operatorname{dom}\operatorname{cod} A = (A - \mathsf{B}(A)) - \mathsf{E}(A - \mathsf{B}(A)) = A - (\mathsf{B}(A) \cup \mathsf{E}(A - \mathsf{B}(A))),$$

so it suffices to show that $\mathsf{E}(A) \cup \mathsf{E}(A - E(A)) = \mathsf{B}(A) \cup \mathsf{E}(A - \mathsf{B}(A))$, which is clear in dimensions greater than or equal to $\dim A = n$ say.

'$\subset$' Suppose $x \in \mathsf{E}(A)$, x of dimension $k<n$. If $x \in \mathsf{B}(A)$, then $x \in$ RHS so suppose $x \notin \mathsf{B}(A)$. By pasting scheme condition (iv) there exists v_0 with $v_0 \mathsf{E}_k^{n-1} x$ (see diagram overleaf). If $v_0 \in A - \mathsf{B}(A)$, then $x \in$ RHS. If $v_0 \notin A - \mathsf{B}(A)$, then there must be a w_0 with $w_0 \mathsf{B}_{n-1}^n v_0 \mathsf{E}_k^{n-1} x$ whence by the *dual$_n$* of condition (v) there exists $v_1 \in \mathsf{E}(w_0)$ with $v_1 \mathsf{E}_k^{n-1} x$. Repeating, we eventually obtain $v_m \in A - \mathsf{B}(A)$ and $x \in \mathsf{E}(v_m)$.



Suppose $x \in \mathsf{E}(A - \mathsf{E}(A))$, say $v_0 \mathsf{E}_k^{n-1} x$ with $v_0 \in A - \mathsf{E}(A)$ and suppose $x \notin \mathsf{B}(A)$ (second diagram above). As before, if $v_0 \in A - \mathsf{B}(A)$, then $x \in$ RHS; otherwise we apply condition (v) until we obtain some $v_m \in A - \mathsf{B}(A)$ with $x \in \mathsf{E}(v_m)$.

'$\supset$' The converse inclusion is precisely the *dual*$_n$ of the above. □

3. Well-formed pasting schemes

We have shown that finite pasting schemes with no direct loops have sensible notions of domain and codomain which satisfy the basic equation dom dom $\doteq$ dom cod. If a finite pasting scheme parametrizes a composable diagram, then its highest dimensional elements must agree in orientation. In this section we describe *well-formed* pasting schemes – those in which the arrangements of the highest dimensional cells in the scheme, and in all of its domains and codomains, are *compatible*.

If A is a finite k-dimensional pasting scheme with no direct loops, write

$$s_n(A) = \begin{cases} A & \text{if } n \geq k, \\ \operatorname{dom}^{k-n} A & \text{if } n < k, \end{cases} \qquad t_n(A) = \begin{cases} A & \text{if } n \geq k, \\ \operatorname{cod}^{k-n} A & \text{if } n < k. \end{cases}$$

Notice that if $n < k$, then $s_n(A)$ and $t_n(A)$ are n-dimensional by Lemma 2. We call $s_n(A)$ the *n-source* of A, and $t_n(A)$ the *n-target* of A.

A pasting scheme A of dimension $k > 0$ is called *compatible* when for any $x, y \in A_k$, if $x \neq y$, then $\mathsf{B}_{k-1}(x) \cap \mathsf{B}_{k-1}(y) = \emptyset$ and $\mathsf{E}_{k-1}(x) \cap \mathsf{E}_{k-1}(y) = \emptyset$. A zero-dimensional pasting scheme is called compatible if it is a singleton.

A subgraded set X of a pasting scheme A is called a *subpasting scheme* of A if $y \in \mathsf{R}(X)$ implies $y \in X$.

A finite pasting scheme A with no direct loops is called *well formed* if

(i) A is compatible;

(ii) for all $n \geq 0$, both $s_n(A)$ and $t_n(A)$ are compatible subpasting schemes of A.

Example 4. (i) The pasting scheme of Example 1 is well formed.

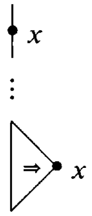
(ii) Any finite chain of abutting arrows (head to tail and without loops) represents a well-formed pasting scheme.

(iii) All the diagrams involving 2-cells in [6] may be expressed as well-formed pasting schemes or assert the equality of two subdiagrams which may be expressed as well-formed pasting schemes.

Examples of well-formed pasting schemes of dimension greater than two appear in [5].

4. Loop-free pasting schemes

Pasting schemes with no direct loops (and even well-formed pasting schemes with no direct loops), may still exhibit subtle looping behaviour like



where the lines should be thought of as k-dimensional, the double arrow as $k+1$ dimensional, x as $k-1$ dimensional, and the ellipses $(\cdots)$ as j-dimensional with $j<k$. In this section we set down the conditions (again rather technical) which eliminate such behaviour. Schemes satisfying these conditions are called *loop free* and in the remainder of this work we show that loop-free schemes and well-formed subschemes of them, behave as we expect pasting schemes should.

A pasting scheme B is called *loop free* if

(i) B has no direct loops;

(ii) for any $x \in B$, $\mathsf{R}(x)$ is well formed;

(iii) for any $k-1$ dimensional well-formed subpasting scheme A of B and any $x \in B_k$ with $\operatorname{dom} \mathsf{R}(x) \subset A$,

(a) $A \cap \mathsf{E}(x) = \emptyset$;

(b) if $y \in A$ and $\mathsf{B}(x) \cap \mathsf{R}(y) \neq \emptyset$, then $y \in \mathsf{B}(x)$;

(iv) for any well-formed j-dimensional subpasting scheme A of B and any $x \in B$ with $s_j(\mathsf{R}(x)) \subset A$, if $u, u' \in s_j(\mathsf{R}(x))$ and, for some $v \in A_j$, $u \lhd_A v \lhd_A u'$, then $v \in s_j(\mathsf{R}(x))$

and dually.

Remark 5. In [4] the author shows that condition (iii) is a consequence of the other three conditions. For now we include all four conditions because the presence of condition (iii) greatly simplifies the development of the theory.

Example 6. All the well-formed pasting schemes of Example 4 are loop free.

From now on we will consider only loop-free pasting schemes. In this and the next section we establish some of their properties.

Proposition 7. *Suppose B is a loop-free pasting scheme, $x \in B_k$, then*

$$\operatorname{dom} \mathsf{R}(x) = \mathsf{R}(\mathsf{B}_{k-1}(x)).$$

Proof. Firstly, $\operatorname{dom} \mathsf{R}(x) = \mathsf{R}(x) - \mathsf{E}(x) \subset \mathsf{R}(\mathsf{B}_{k-1}(x))$ because, using pasting scheme condition (iv), $\mathsf{R}(x) = \mathsf{R}(\mathsf{B}_{k-1}(x)) \cup \mathsf{R}(\mathsf{E}_{k-1}(x)) \cup \{x\}$ and, using pasting scheme condition (v), $\mathsf{R}(\mathsf{E}_{k-1}(x)) \cup \{x\} \subset \mathsf{R}(\mathsf{B}_{k-1}(x)) \cup \mathsf{E}(x)$. But, since B has no direct loops, $\mathsf{B}_{k-1}(x) \subset \mathsf{R}(x) - \mathsf{E}(x) = \operatorname{dom} \mathsf{R}(x)$ and so $\mathsf{R}(\mathsf{B}_{k-1}(x)) \subset \mathsf{R}(\operatorname{dom} \mathsf{R}(x)) = \operatorname{dom} \mathsf{R}(x)$ by loop free condition (ii). □

Proposition 8 (Pasting on). *Suppose B is a loop-free pasting scheme. If A is a well-formed $(k-1)$-dimensional subpasting scheme of B and $x \in B_k$ satisfies* $\operatorname{dom} \mathsf{R}(x) \subset A$, *then $A \cup \mathsf{R}(x)$ is a well-formed subpasting scheme of B.*

Proof. The scheme $A \cup \mathsf{R}(x)$ has only a single k-dimensional element and so is compatible. Furthermore,

$$\begin{aligned} s_{k-1}(A \cup \mathsf{R}(x)) &= A \cup \mathsf{R}(x) - \mathsf{E}(x) = (A - \mathsf{E}(x)) \cup (\mathsf{R}(x) - \mathsf{E}(x)) \\ &= A \cup \operatorname{dom} \mathsf{R}(x) = A \end{aligned}$$

which is well formed. Hence for all $n \neq k$, $s_n(A \cup \mathsf{R}(x))$ is well formed. Furthermore, since for $j < k-1$

$$t_j(A \cup \mathsf{R}(x)) = t_j(t_{k-1}(A \cup \mathsf{R}(x))) = t_j(s_{k-1}(A \cup \mathsf{R}(x))),$$

$t_n(A \cup \mathsf{R}(x))$ is well formed for all $n \neq k, k-1$.

It remains only to consider

$$\begin{aligned} t_{k-1}(A \cup \mathsf{R}(x)) &= A \cup \mathsf{R}(x) - \mathsf{B}(x) = (A - \mathsf{B}(x)) \cup (\mathsf{R}(x) - \mathsf{B}(x)) \\ &= (A - \mathsf{B}(x)) \cup \operatorname{cod} \mathsf{R}(x) \end{aligned}$$

which is a subpasting scheme since $\operatorname{cod} \mathsf{R}(x)$ and (using loop free condition (iii)) $A - \mathsf{B}(x)$ are subpasting schemes. Finally, $t_{k-1}(A \cup \mathsf{R}(x))$ is compatible since suppose not, then there exists $z, w \in (t_{k-1}(A \cup \mathsf{R}(x)))_{k-1}$, $z \neq w$, such that there exists $a \in \mathsf{D}_{k-2}(z) \cap \mathsf{D}_{k-2}(w)$, $\mathsf{D} = \mathsf{E}$ or $\mathsf{D} = \mathsf{B}$. Now z, w are not both in $A - \mathsf{B}(x)$, since if it is $(k-1)$-dimensional, then it must be compatible being a subpasting scheme of a compatible $(k-1)$-dimensional pasting scheme, nor in $\operatorname{cod} \mathsf{R}(x)$ since it is compatible. Hence, without loss of generality, suppose $w \in A - \mathsf{B}(x)$ and $z \in \operatorname{cod} \mathsf{R}(x)$. Now $a \notin \mathsf{E}(x)$ since $a \in \mathsf{D}(w) \subset A$ and $A \cap \mathsf{E}(x) = \emptyset$ so, by pasting scheme condition (v), there exists $v \in \mathsf{B}_{k-1}(x)$ with $a \in \mathsf{D}_{k-2}(v)$ contradicting the compatibility of A. □

Theorem 9. *Suppose that Q is a loop-free pasting scheme and that A, B are well-formed subpasting schemes of Q with $s_n(B) = t_n(A)$, then $A \cap B = s_n(B)$.*

Proof. By induction over the dimension of $A \cup B$.

If $A \cup B$ is of dimension less than or equal to n, then $s_n(B) = t_n(A)$ implies that

$A = t_n(A) = s_n(B) = B = A \cap B$ as required.

Suppose $A \cup B$ is of dimension $n+1$ and $x \in A \cap B$ but $x \notin s_n(B)$, then A, B are both $(n+1)$-dimensional since otherwise $t_n(A) = A$ or $s_n(B) = B$, whence $x \notin s_n(A) = t_n(B)$ implies $x \notin A \cap B$. Thus $s_n(B) = \operatorname{dom} B$ and $x \notin s_n(B)$ implies $x \in \mathsf{E}(w)$ for some $w \in B_{n+1}$.

In B, let $Y = \lhd_B(w) = \{y \in B\colon y \lhd w\}$. We show that there is an enumeration $y_0, y_1, \ldots, y_r$ of the elements of Y such that

$$\mathsf{B}_n(y_i) \subset \operatorname{dom} B \cup \mathsf{E}(\{y_j\colon j<i\}) - \mathsf{B}(\{y_j\colon j<i\}).$$

Firstly, there exists a suitable y_0 since, for any $y \in Y$, if $\mathsf{B}_n(y) \not\subset \operatorname{dom} B$ it can only be because there is some $y' \lhd y$ with $\mathsf{E}_n(y') \cap \mathsf{B}_n(y) \neq \emptyset$. Repeating this, we obtain $y'' \lhd y' \lhd y$ etc. Since B is finite and has no direct loops, this process must terminate yielding some suitable $y^{(q)} = y_0$ say. Similarly, there exists $y_1 \in Y - \{y_0\}$ such that

$$\mathsf{B}_n(y_1) \subset \operatorname{dom} B \cup \mathsf{E}(y_0)$$

etc. Furthermore, because of the compatibility of Y (inherited from B), if

$$\mathsf{B}_n(y_i) \subset \operatorname{dom} B \cup \mathsf{E}(\{y_j\colon j<i\}),$$

then

$$\mathsf{B}_n(y_i) \subset \operatorname{dom} B \cup \mathsf{E}(\{y_j\colon j<i\}) - \mathsf{B}(\{y_j\colon j<i\}).$$

Now since $\operatorname{dom} \mathsf{R}(y_0) = \mathsf{R}(\mathsf{B}_n(y_0))$ (Proposition 7) we can apply Proposition 8 to conclude that

$$\operatorname{cod}(\operatorname{dom} B \cup \mathsf{R}(y_0)) = \operatorname{dom} B \cup \mathsf{R}(y_0) - \mathsf{B}(y_0) = \operatorname{dom} B \cup \mathsf{E}(y_0) - \mathsf{B}(y_0)$$

is well formed. Proceeding inductively,

$$B' = \operatorname{dom} B \cup \mathsf{E}(Y) - \mathsf{B}(Y)$$

is well formed.

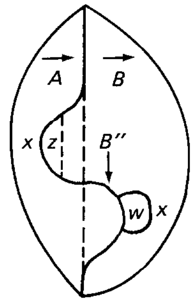
Similarly, $x \notin t_n(A)$ implies $x \in \mathsf{B}(z)$ for some $z \in A_{n+1}$ and if

$$z \lhd_A z' \quad \text{and} \quad v \in \mathsf{E}_n(z') \cap \operatorname{cod} A = \mathsf{E}_n(z') \cap \operatorname{dom} B,$$

then $v \in B'$ since $v \in \mathsf{B}_n(y)$ for some $y \in Y$ would give a direct loop. Thus

$$B'' = B' \cup \mathsf{B}(\rhd_A(z) \cup \{z\}) - \mathsf{E}(\rhd_A(z) \cup \{z\})$$

is well formed as above. But $\operatorname{dom} \mathsf{R}(w) \subset B''$, and $x \in \mathsf{E}(w) \cap B''$ which contradicts Q being loop free.



Now suppose $h>n+1$ and that for all well-formed A, B with $A\cup B$ of dimension less than h and $s_n(B)=t_n(A)$ we have $A\cap B=s_n(B)$. Let A, B be well-formed subpasting schemes of Q with $A\cup B$ of dimension h. Once again, suppose $x\in A\cap B$ but $x\notin s_n(B)$. We may suppose x to be of dimension less than h since, if not, choose any $v\in \mathsf{E}_{h-1}(x)$, then $v\in A\cap B$, v is of dimension $h-1$ and $v\notin s_n(B)$, so v will do. Let $P=\{a\in A_h\colon x\in \mathsf{E}(a)\}$ and $Q=\{b\in B_h\colon x\in \mathsf{E}(b)\}$. Put

$$A' = s_{h-1}(A)\cup \mathsf{E}(\lhd_A P\cup P)-\mathsf{B}(\lhd_A P\cup P),$$
$$B' = s_{h-1}(B)\cup \mathsf{E}(\lhd_B Q\cup Q)-\mathsf{B}(\lhd_B Q\cup Q).$$

Then A', B' are well formed and $s_n(B')=s_n(B)=t_n(A)=t_n(A')$ but $x\in A'\cap B'$, $x\notin s_n(B')=t_n(A')$ and A', B' are of dimension less than h, contradicting the inductive hypothesis. □

5. Paring down well-formed schemes

A pasting scheme A of dimension $k>0$ is called *strongly compatible* when, for any $x, y\in A_k$, $x\neq y$ implies $\mathsf{B}(x)\cap\mathsf{B}(y)=\emptyset$ and $\mathsf{E}(x)\cap\mathsf{E}(y)=\emptyset$. A zero-dimensional pasting scheme is strongly compatible if it is a singleton.

Proposition 10. *Every loop-free well-formed pasting scheme is strongly compatible.*

Proof. Suppose that Q is a k-dimensional loop-free well-formed, pasting scheme. By way of contradiction, suppose $w, z\in A_k$, $w\neq z$, and $a\in\mathsf{E}(w)\cap\mathsf{E}(z)$. Let $Y=\lhd\{w,z\}$.

As in the proof of Theorem 9, $\operatorname{dom} Q\cup\mathsf{E}(Y)-\mathsf{B}(Y)=A$ say, is well formed. Furthermore, either $\mathsf{B}_{k-1}(w)\subset A$ and $\mathsf{B}_{k-1}(z)\subset A$ and hence $\operatorname{dom}\mathsf{R}(w)\subset A$ so, using Proposition 8, $A'=A\cup\mathsf{E}(w)-\mathsf{B}(w)$ is well formed, or $w\in Y$ or $z\in Y$ (but not both); suppose without loss of generality $w\in Y$ and then let $A'=A$. Now in either case $a\in A'$ (since $a\in\mathsf{E}(w)$ and, because Q has no direct loops, for any $y\in Y\cup\{w\}$, $a\notin\mathsf{B}(y)$) and $\mathsf{B}_{k-1}(z)\subset A'$, hence $\operatorname{dom}\mathsf{R}(z)\subset A'$, but $a\in\mathsf{E}(z)$ contradicting Q being loop free. □

Proposition 11 (Paring down). *Suppose that Q is k-dimensional, loop free and well formed, and $y\in Q_k$ satisfies* $\operatorname{dom}\mathsf{R}(y)\subset\operatorname{dom} Q$, *then* $Q-\mathsf{B}(y)$ *is well formed.*

Proof. If $Q-\mathsf{B}(y)$ is $(k-1)$-dimensional, then $Q-\mathsf{B}(y)=\operatorname{cod} Q$ which is well formed, so suppose $Q-\mathsf{B}(y)$ is k-dimensional. Then $Q-\mathsf{B}(y)$ is compatible since Q is, and it is a subpasting scheme since, if not, then there exists $a\in\mathsf{B}(y)\cap\mathsf{R}(z)$ for some $z\in Q_k-\{y\}$. Furthermore, $a\notin\mathsf{E}(z)$ since $a\in\mathsf{B}(y)\subset\operatorname{dom} Q$, therefore $a\in\mathsf{R}(w)$ for some $w\in\mathsf{B}_{k-1}(z)$ whence $w\in\operatorname{dom} Q$ or $\mathsf{E}(z_2)$ etc. to obtain $w\in\operatorname{dom} Q$ with $a\in\mathsf{R}(w)$, $w\in\mathsf{B}_{k-1}(z_r)$; but then Q being loop free implies $w\in\mathsf{B}_{k-1}(y)$ and $z_r\neq y$

(because $a \notin \mathsf{R}(\mathsf{E}_{k-1}(y))$ but $a \in \mathsf{R}(\mathsf{E}_{k-1}(z_r))$) which contradicts the compatibility of Q.

Furthermore,

$$\begin{aligned}\operatorname{cod}(Q-\mathsf{B}(y)) &= Q-\mathsf{B}(y)-\mathsf{B}(Q_k-\{y\})\\ &= Q-(\mathsf{B}(y)\cup \mathsf{B}(Q_k-\{y\}))\\ &= Q-\mathsf{B}(Q_k)=\operatorname{cod} Q\end{aligned}$$

is well formed and so $s_n(Q-\mathsf{B}(y))$, $t_n(Q-\mathsf{B}(y))$ are well formed for all $n\neq k, k-1$.

It remains only to show that $s_{k-1}(Q-\mathsf{B}(y))=\operatorname{dom}(Q-\mathsf{B}(y))$ is a compatible pasting scheme. Now,

$$\begin{aligned}\operatorname{dom}(Q-\mathsf{B}(y)) &= Q-\mathsf{B}(y)-\mathsf{E}(Q_k-\{y\})\\ &= Q-\mathsf{E}(Q_k-\{y\})-\mathsf{B}(y)\\ &= Q-\mathsf{E}(Q_k)\cup \mathsf{E}(y)-\mathsf{B}(y) \quad \text{(Proposition 10)}\\ &= \operatorname{dom} Q\cup \mathsf{E}(y)-\mathsf{B}(y)\\ &= \operatorname{cod}(\operatorname{dom} Q\cup \mathsf{R}(y))\end{aligned}$$

which is a compatible pasting scheme by Proposition 8. □

6. Categories of pasting schemes

Well-formed pasting schemes parametrize 'composable' diagrams. If, in a loop-free pasting scheme, we have two well-formed subpasting schemes whose n-source and n-target match up, we should be able to paste them together to obtain another well-formed scheme. This is made precise in the following theorem. (For elementary definitions of ω-categories and free ω-categories the reader is referred to [10].)

Theorem 12. *Suppose that S is a loop-free pasting scheme and $\mathcal{P}$ the collection of well-formed subpasting schemes of S, then $(\mathcal{P}, (s_i, t_i, \cup)_{i\in\omega})$ is an ω-category.*

Proof. The elementary properties of s_i and t_i follow from their definition in terms of dom and cod and Theorem 3; identity, associativity and the middle four interchange laws follow from analogous properties of union (for identity $A\subset B$ implies $A\cup B=B$).

For the other composition axioms suppose $s_i(B)=t_i(A)$ for some $A, B\in\mathcal{P}$. We prove by induction over the dimension of $A\cup B$ that

(a) $s_i(A\cup B)=s_i(A)$,

(b) $s_j(A\cup B)=s_j(A)\cup s_j(B)$ for $j>i$, and

(c) $A\cup B$ is well formed.

If $A\cup B$ is of dimension less than or equal to i, then $s_i(B)=t_i(A)$ implies $A=B$ so (a), (b) and (c) follow.

Suppose A, B are well formed, $A \cup B$ is of dimension $h > i$ and that for all well-formed A, B with $A \cup B$ of dimension less than h, $s_i(B) = t_i(A)$ implies (a), (b) and (c).

(a) If $h > i+1$, then

$$\begin{aligned} s_i(A \cup B) &= s_i(s_{h-1}(A \cup B)) = s_i(\mathrm{dom}(A \cup B)) \\ &= s_i((A \cup B) - \mathsf{E}(A \cup B)) \\ &= s_i((A - \mathsf{E}(A_h) - \mathsf{E}(B_h)) \cup (B - \mathsf{E}(B_h) - \mathsf{E}(A_h))) \\ &= s_i((s_{h-1}(A) - \mathsf{E}(B_h)) \cup (s_{h-1}(B) - \mathsf{E}(A_h))) \\ &= s_i(s_{h-1}(A) \cup s_{h-1}(B)) \quad (\text{since } A \cap B = s_i(B)) \\ &= s_i(s_{h-1}(A)) \quad (\text{by inductive hypothesis (a)}) \\ &= s_i(A). \end{aligned}$$

If $h = i+1$, then

$$\begin{aligned} s_i(A \cup B) &= s_i((s_{h-1}(A) - \mathsf{E}(B_h)) \cup (s_{h-1}(B) - \mathsf{E}(A_h))) \\ &= s_i(A) \quad (\text{since } s_i(B) - \mathsf{E}(A_h) \subset s_i(A)). \end{aligned}$$

(b) If $j \geq h$, then

$$s_j(A \cup B) = A \cup B = s_j(A) \cup s_j(B).$$

If $j < h$, then

$$\begin{aligned} s_j(A \cup B) &= s_j(s_{h-1}(A \cup B)) = s_j(\mathrm{dom}(A \cup B)) \\ &= s_j(s_{h-1}(A) \cup s_{h-1}(B)) \quad (\text{as above}) \\ &= s_j(s_{h-1}(A)) \cup s_j(s_{h-1}(B)) \quad (\text{inductive hypothesis (b)}) \\ &= s_j(A) \cup s_j(B). \end{aligned}$$

(c) $A \cup B$ is a compatible pasting scheme since A and B are and $\mathsf{E}^h(A) \cap \mathsf{E}^h(B) = \emptyset$ and $\mathsf{B}^h(A) \cap \mathsf{B}^h(B) = \emptyset$ (Theorem 9). Furthermore, if $h > i+1$, then $\mathrm{dom}(A \cup B) = s_{h-1}(A) \cup s_{h-1}(B)$ (by (b)), while if $h = i+1$, then $\mathrm{dom}(A \cup B) = s_i(A)$ (by (a)). In either case $\mathrm{dom}(A \cup B)$ is well formed. Similarly, using dual forms of (a) and (b), $\mathrm{cod}(A \cup B)$ is well formed. □

If S is a loop-free pasting scheme, then the ω-category of Theorem 12 is called the *ω-category of components of S*. The pasting theorem, which asserts that all strategies for composing cells in a 'composable diagram' in an ω-category yield the same result, follows from the freeness of ω-categories of components.

Theorem 13. *Suppose S is a loop-free pasting scheme, then the ω-category of components of S is the free ω-category generated by the* $\mathsf{R}(x)$, $x \in S$.

Proof. The fact that the ω-category of components of S is generated by the $\mathsf{R}(x)$, $x \in S$, follows by induction.

Suppose that A is a well-formed subpasting scheme of S of dimension k, $x \in A_k$

and $A \neq \mathsf{R}(x)$. We show that for some j there exists $y \in A_j$, $y \notin \mathsf{R}(x)$, with either $A - \mathsf{B}(y)$ and $s_{j-1}(A) \cup \mathsf{R}(y)$ well formed and $j-1$ composable with composite A, or $A - \mathsf{E}(y)$ and $t_{j-1}(A) \cup \mathsf{R}(y)$ well formed and $j-1$ composable with composite A.

Since $A \neq \mathsf{R}(x)$, there exists y_0 of maximum dimension say j, such that $y_0 \notin \mathsf{R}(x)$. Furthermore, by part (iv) of the definition of loop-free,

$$\text{either } \lhd_A\{y_0\} \cap \mathsf{R}(x) = \emptyset, \text{ or } \rhd_A\{y_0\} \cap \mathsf{R}(x) = \emptyset.$$

Suppose $\lhd_A\{y_0\} \cap \mathsf{R}(x) = \emptyset$ (the other case follows dually), then any $\lhd_A$-minimal element of $\lhd_A\{y_0\}$ will do for y since $s_{j-1}(A) \cup \mathsf{R}(y)$ is well formed (Proposition 8) and $A - \mathsf{B}(y)$ is well formed (Proposition 11, Proposition 8 and Theorem 12).

Freeness follows exactly as in [10, Theorem 18]. □

Remark 14. It is noteworthy that, despite our different context and greater generality, Street's proof [10, Theorem 18] generalizes with only notational modifications to our Theorem 13.

7. The pasting theorem

We have described loop-free pasting schemes which are the appropriate parametrizing objects for diagrams in ω-categories. Among these are the well-formed loop-free pasting schemes which are the appropriate parametrizing objects for composable diagrams. It remains to describe the parametrizations themselves and to establish that a well-formed loop-free pasting scheme which is the domain of a given parametrization in some ω-category, determines a unique cell called the *composite* of the diagram in the ω-category. These two tasks are interwoven. We proceed inductively.

If C is an ω-category write C_i for the set of i-cells of C and $|C|_i$ for the i-category consisting of the i-cells of C with the first i compositions.

A *realization* (A, f_i) of a pasting scheme A in an ω-category C is a collection of functions $f_i : A_i \to C_i$, $i = 0, 1, \ldots$, which we will sometimes view as functions $f_i : A_i \to C$, into the underlying set of the ω-category C.

We write $\mathscr{P}(A)$ for the ω-category of components of A – its elements are well-formed subpasting schemes of A (Theorem 12). The j-category $|\mathscr{P}(A)|_j$ is the sub-ω-category of $\mathscr{P}(A)$ whose elements are well-formed subpasting schemes of A of dimension less than or equal to j. For each k we have a function $\mathsf{R}(\ ) : A_k \to |\mathscr{P}(A)|_k$ and functors (which we will not name) including $|\mathscr{P}(A)|_j$ in $|\mathscr{P}(A)|_k$, $j < k$. A realization (A, f_i) is said to be *n-extendable* when there exists a unique functor $f : |\mathscr{P}(A)|_n \to C$ such that the diagrams (of functions)

$$\begin{array}{ccc} |\mathscr{P}(A)|_k & \longrightarrow & |\mathscr{P}(A)|_n \\ {\scriptstyle \mathsf{R}(\ )}\big\uparrow & & \big\downarrow{\scriptstyle f} \\ A_k & \xrightarrow[f_k]{} & C \end{array}$$

commute for all $k \leq n$.

Inductive definition. Any realization (A, f_i) will be called *zero-appropriate*, and is zero-extendable (f_0 is already a functor $|\mathscr{P}(A)|_0 \to C$). Suppose that every n-appropriate realization is n-extendable and suppose given an n-appropriate realization (A, f_i). Then we have

$$\begin{array}{ccc} A_{n+1} & & \\ \big\downarrow \mathsf{R} & \searrow^{f_{n+1}} & \\ |\mathscr{P}(A)|_{n+1} & \dashrightarrow & C \\ s_n \Big\downarrow\Big\downarrow t_n & & s_n \Big\downarrow\Big\downarrow t_n \\ |\mathscr{P}(A)|_n & \xrightarrow[f]{} & C \end{array}$$

We say that (A, f_i) is $(n+1)$*-appropriate* if $s_n f_{n+1} = f s_n \mathsf{R}$ and $t_n f_{n+1} = f t_n \mathsf{R}$, whence, by the freeness of $\mathscr{P}(A)$, (A, f_i) is $(n+1)$-extendable. A realization is called *appropriate* if it is n-appropriate for all n.

Thus a realization is nothing more than an assignment, to each n-dimensional element of a pasting scheme, of an n-cell in an ω-category. A realization is appropriate if it respects s_k and t_k – if it is categorically sensible. A *diagram* (A, f_i) in an ω-category C is a loop-free pasting scheme A together with an appropriate realization $f_i : A_i \to C_i$. A *composable diagram* in an ω-category C is a well-formed loop-free pasting scheme A together with an appropriate realization $f_i : A_i \to C_i$.

Furthermore, we have shown, using the freeness of $\mathscr{P}(A)$, that appropriate realizations are n-extendable for all n. Thus if (A, f_i) is a composable diagram in C, then there is a unique functor $f : \mathscr{P}(A) \to C$ which extends all the f_i (since $|\mathscr{P}(A)|_k = \mathscr{P}(A)$ for $k = \dim A < \infty$). This establishes the bijection between appropriate realizations of A in C and functors $\mathscr{P}(A) \to C$ referred to in Section 1.

Observation 15 (The pasting theorem). If A is an n-dimensional, well-formed loop-free pasting scheme then $A \in |\mathscr{P}(A)|_n$. Furthermore, appropriate realizations are extendable. Thus a composable diagram (A, f_i) in an ω-category C determines uniquely, via $f(A)$, a cell of C called the *composite* of (A, f_i) (where f is the functor corresponding to the realization (f_i)).

Remark 16. Two aspects of the pasting theorem deserve further comment:

(i) It has long been assumed that the pasting theorem would be proved by an inductive argument over the structure of the diagram. This is indeed the case here but the simple induction, which occurs in the proof of the freeness part of Theorem 13, has been omitted.

(ii) The above inductive definition may seem surprisingly complicated. The complications arise because the appropriateness of a realization of an $n+1$ dimensional pasting scheme depends on the n-category pasting theorem – one must check that the n-source and n-target of each element of the scheme are sent to the n-source and n-target of the realization of the element. But it is the n-category pasting theorem which says that the sources and targets are sent to well-defined cells, and the n-category pasting theorem is equivalent to the statement "n-appropriate realizations are n-extendable".

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